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The Spectrum of the Γ-Invariant of a Bilinear Space

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To every symmetric bilinear space (X, ϕ) of regular uncountable dimension κ, an invariant $\Gamma(X, \phi) \in \mathscr{P}(\kappa)/\mathscr{F}(\kappa)$ (where $\mathscr{F}(\kappa)$ is the club filter) can be assigned. We prove that in dimension $\aleph_2$ the spectrum of Γ cannot be determined in *ZFC*. For this, on the one hand we show that under *CH*, Γ attains the maximal (with respect to a restriction provable in *ZFC*) spectrum; we also show that *CH* is not necessary for this result. On the other hand we show that in a variation of Mitchell's model, which is obtained by collapsing a weakly compact cardinal to ω_2, the spectrum of Γ in dimension $\aleph_2$ is much thinner than the maximal one.

INTRODUCTION

Throughout this paper, by a symmetric bilinear space, or just bilinear space for short, we mean a vector space X over some arbitrary field k which is endowed with a symmetric bilinear form ϕ: $X \times X \to k$, i.e., ϕ is linear in both arguments and $\phi(x, y) = \phi(y, x)$ always. We do *not* require $\phi(x, x) \neq 0$ for $x \neq 0$. As usual we have the notion of *orthogonal*

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complement of a subspace $U \subset X$, defined by $U^{\perp} = \{x \in X\colon \forall y \in U\ (\phi(x, y) = 0)\}$. (X, ϕ) is called *nondegenerate* if $X^{\perp} = \{0\}$.

In [Ek1, Ek2] Eklof introduced the Γ-invariant for abelian groups which measures the degree of freeness. In [Ap], inspired by Eklof's work, Appenzeller introduced the Γ-invariant for a bilinear space (X, ϕ) of some regular uncountable dimension κ. If $\kappa = \aleph_1$, then $\Gamma(X, \phi)$ measures how close (X, ϕ) is to having an orthogonal basis; if $\kappa > \aleph_1$, then $\Gamma(X, \phi)$ measures how close (X, ϕ) is to being orthogonally decomposable into subspaces of dimension $< \kappa$. $\Gamma(X, \phi)$ is an element of the Boolean algebra $\mathscr{P}(\kappa)/\mathscr{F}(\kappa)$ where $\mathscr{F}(\kappa)$ is the club filter on κ, that is, the filter generated by the closed unbounded subsets of κ. By the *spectrum* of Γ we mean the range of Γ, for some fixed dimension. Appenzeller proved in [Ap] (using ideas from [Sh]) that in dimension $\aleph_1$ the spectrum of Γ is full, that is, all of $\mathscr{P}(\omega_1)/\mathscr{F}(\omega_1)$. In [Sp] it was observed that this is false for $\kappa > \omega_1$. If $e \in \mathscr{P}(\kappa)/\mathscr{F}(\kappa)$ belongs to the spectrum of Γ, then there must exist a representative $S \in e$ which is *heavy* in the sense of Definition 1.5 below. In this paper we show that this is the only restriction provable in *ZFC*. In Section 2 we prove that if the continuum hypothesis holds, for every heavy $S \subset \omega_2$ and every field k there exists a bilinear space over k which has S as its Γ-invariant. For this we isolate a combinatorial principle (existence of a *coherent ladder system*) which resembles Jensen's $\square_{\omega_1}$-principle and holds under *CH*.

In Section 3 we prove that *CH* is not necessary for the result in Section 2. We prove that if $\square_{\omega_1}$ holds and for every sequence $\langle C_\alpha\colon \alpha < \omega_2 \rangle$ of clubs in ω_1 there exists a club which is eventually included in every C_α, then there exist coherent ladder systems. Since these two properties can be forced and are preserved under *ccc* forcings we can obtain the desired result.

In Section 4 we show that the results of Sections 2 and 3 cannot be proved in *ZFC*. Assuming the existence of a weakly compact cardinal we construct a model where the spectrum of Γ in dimension $\aleph_2$ is much thinner than the maximal possible one in the sense explained above. The model is due to Mitchell (see [M]). It is obtained (roughly) by collapsing the weakly compact λ to ω_2 by forcing λ times a Cohen real followed by a Cohen subset of ω_1. We show that in this model, for no stationary costationary S belonging to the weakly compact filter of λ (in the ground model V) and for no field k (in the extension) there exists a bilinear space with S as its Γ-invariant.

We do not know whether large cardinals are necessary for this result. Our set-theoretic notation is standard. If λ is a regular cardinal, by $\operatorname{cof}(\lambda)$ we denote the class of ordinals with cofinality λ. By succ, lim we denote the class of successor, limit ordinals, respectively.

We think that our results exemplify clearly how a natural algebraic question may be attacked and solved using sophisticated set-theoretic methods such as infinitary combinatorics. See [K] for an introduction to these techniques.

1. THE Γ-INVARIANT OF A BILINEAR SPACE

Let k be a field, X a vector space over k, and $\phi\colon X \times X \to k$ a symmetric bilinear form which is nondegenerate. We assume that the dimension of X is uncountable and regular. Let $\kappa = \dim X$. A *κ-filtration* of X is a family $\langle X_\alpha\colon \alpha < \kappa \rangle$ of subspaces of X with the following properties:

(1) $\dim X_\alpha < \kappa$,

(2) $\alpha < \beta \Rightarrow X_\alpha \subset X_\beta$,

(3) $\lim(\lambda) \Rightarrow X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha$,

(4) $X = \bigcup_{\alpha < \kappa} X_\alpha$.

The following fact is easy to verify:

FACT 1.1. *Given two κ-filtrations $\langle X_\alpha\colon \alpha < \kappa \rangle$ and $\langle Y_\alpha\colon \alpha < \kappa \rangle$ of X, then the set $\{\alpha < \kappa\colon X_\alpha = Y_\alpha\}$ is a club.*

DEFINITION 1.2. We define $\Gamma(X, \phi)$ to be the equivalence class of $\{\alpha < \kappa\colon X_\alpha \oplus X_\alpha^\perp \neq X\}$ in the Boolean algebra $\mathscr{P}(\kappa)/\mathscr{F}(\kappa)$ where $\mathscr{F}(\kappa)$ is the club filter. By Fact 1.1, $\Gamma(X, \phi)$ does not depend upon $\langle X_\alpha\colon \alpha < \kappa \rangle$. Later we will often work with representatives for equivalence classes instead of the equivalence classes themselves, and so $\Gamma(X, \phi)$ will be considered as a subset of κ.

FACT 1.3. *If $X = X_\alpha + X_\alpha^\perp$, then $X = X_\alpha \oplus X_\alpha^\perp$.*

Proof. Let $x \in X_\alpha \cap X_\alpha^\perp$ and $y \in X$. By assumption, $y = y_1 + y_2$ where $y_1 \in X_\alpha$ and $y_2 \in X_\alpha^\perp$. Since $x \in X_\alpha$ we have $x \perp y_2$, i.e., $\phi(x, y_2) = 0$, since $x \in X_\alpha^\perp$ we have $x \perp y_1$, hence $x \perp y$. Since y was arbitrary, by nondegeneracy of X we conclude $x = 0$. ∎

In [Ap], for every $S \subset \omega_1$ and every field k, a space (X, ϕ) over k of dimension $\aleph_1$ has been constructed with $\Gamma(X, \phi) = S$. For larger dimensions this is not possible. The following lemma is proved by using Fodor's Lemma and Fact 1.3.

LEMMA 1.4 [Sp, Lemma 2, p. 122]. *Let (X, ϕ) be a bilinear space over k of regular dimension $\kappa > \omega_1$, and let $\langle X_\alpha\colon \alpha < \kappa \rangle$ be a κ-filtration. If $\mu \in \Gamma(X, \phi)$ and $\mathrm{cf}(\mu) \geq \omega_1$, then $\Gamma(X, \phi) \cap \mu$ contains a club.*

Proof. By contradiction, suppose that the set $S = \{\alpha < \mu\colon X_\alpha + X_\alpha^\perp\}$ is stationary in μ. By assumption on μ, there exists $x \in X$ such that for no $y \in X_\mu$, $x - y \in X_\mu$. On the other hand, for every $\alpha \in S$ there exists a unique (by Fact 1.3.) $y_\alpha \in X_\alpha$ such that $x - y_\alpha \in X_\alpha^\perp$. Let $f(\alpha)$ be the least β such that $y_\alpha \in X_\beta$. Then f is regressive on the stationary set $S \cap \lim(\mu)$. Hence by Fodor's Lemma there exists $\gamma < \mu$ such that $S' = \{\alpha \in S\colon f(\alpha) < \gamma\}$ is stationary. Now it is easy to see that for $\alpha \in S'$ with $\alpha > \gamma$, y_α is constant, say y. Consequently, $x - y \in X_\alpha^\perp$ for unboundedly many $\alpha < \mu$, and hence $x - y \in X_\mu^\perp$, a contradiction. ∎

DEFINITION 1.5. A set $S \subset \omega_2$ is called *heavy* if S is stationary and for every $\alpha \in S \cap \mathrm{cof}(\omega_1)$, $S \cap \alpha$ contains a club.

The question whether for every heavy $S \subset \omega_2$ there exists a bilinear space with S as its Γ-invariant is not decidable in ZFC, as we will prove in this paper. First we turn to the existence side.

2. FULL SPECTRUM UNDER *CH*

DEFINITION 2.1. For $S \subset \lim(\omega_2)$, a family $\langle s_\alpha\colon \alpha \in S\rangle$ is called a *coherent ladder system on S* if the following properties hold:

(1) $s_\alpha \subseteq \alpha$ is unbounded and o.t. $s_\alpha \le \omega_1$,

(2) $\lim(s_\alpha) \subset S$

(3) $s_\alpha \cap s_\beta$ is an initial segment both of s_α and s_β,

(4) $|\{s_\alpha \cap \beta\colon \alpha \in S\}| = \aleph_1$ for all $\beta < \omega_2$.

THEOREM 2.2. *If there exists a coherent ladder system on S, then for every field k there exists a nondegenerate bilinear space (X, ϕ) over k with $\Gamma(X, \phi) = S$.*

Proof. Let $\langle s_\alpha\colon \alpha \in S\rangle$ be a coherent ladder system on S. Define

$$B = \{s_\alpha \cap \beta\colon \alpha \in S,\ \beta < \omega_2\} \setminus \{0\}.$$

Let X be the vector space over k with B as its algebraic basis. For $x, y \in B$ in order to define $\phi(x, y)$, let λ be the largest limit ordinal $\le$ o.t.$(x \cap y)$. Hence o.t.$(x \cap y) = \lambda + n$ for some $n < \omega$. Now let $\phi(x, y) = n \bmod \mathrm{char}(k)$. Finally expand ϕ linearly on the whole of $X \times X$. First let us check that (X, ϕ) is nondegenerate. Let $\sum_{i<n} a_i x_i \in X$ with $a_i \neq 0$, $i < n$, $n \neq 0$. We have to find $z \in X$ such that $\phi(\sum_{i<n} a_i x_i, z) \neq 0$. Suppose $n = 1$ first. Let λ be the largest limit ordinal such that o.t.$(x_0) = \lambda + m$ for some $m < \omega$. If $m = \mathrm{char}(k)$ (so $2 \le m$) let z be the initial segment of x_0 of length $\lambda + m - 1$. Then $\phi(a_0 x_0, z) \neq 0$. If $m \neq \mathrm{char}(k)$ and

$m \not\equiv 0$ let $z = x_0$. Then clearly $\phi(a_0 x_0, z) \neq 0$. Finally, if $m = 0$ then o.t.(x_0) is a nonzero limit ordinal, since we removed 0 from B. Hence there exists $z \in B$ with $z \subset x_0$ and o.t.(z) is a successor of the form $\mu + l$, where $l \neq \operatorname{char}(k)$. We conclude that $\phi(a_0 x_0, z) \neq 0$.

Hence we may assume $n > 1$. Choose $x \in \{x_i\colon i < n\}$ maximal with respect to inclusion, and let $y = \bigcup\{x \cap x_i\colon x_i \neq x\}$. By (3) and the maximality of x, y is a proper initial segment of x. Let $\gamma = \min(x \setminus y)$ and $y_1 = y \cup \{\gamma\}$. Then $\phi(y, x_i) = \phi(y_1, x_i)$ if $x_i \neq x$, but $\phi(y_1, x) = \phi(y, x) + 1$. Hence if we let $z = y_1 - y$ we conclude $\phi(\sum_{i<n} a_i x_i, z) = a_j \neq 0$, where j is such that $x_j = x$. We have proved that ϕ is nondegenerate.

Next we shall compute the Γ-invariant of (X, ϕ). Define an ω_2-filtration $\langle X_\alpha\colon \alpha < \omega_2 \rangle$ of X as follows: Let X_α be the linear span of the set $\{x \in B\colon \sup(x) < \alpha\}$. Then the axioms of a filtration are easily verified. Axiom 1 follows from property (4) for the coherent ladder system. Now $\Gamma(X, \phi) = S$ will follow from the next two claims.

Claim 1. $\alpha \in S \Rightarrow X_\alpha + X_\alpha^\perp \neq X$.

Proof of Claim 1. It suffices to prove that for no $x \in X_\alpha$, $s_\alpha - x \in X_\alpha^\perp$. Clearly we have $s_\alpha \notin X_\alpha^\perp$, since we may find an (proper) initial segment y of s_α with $y \not\perp s_\alpha$. Then clearly $y \in X_\alpha$. So let $x \in X_\alpha \setminus \{0\}$, say $x = \sum_{i<n} a_i x_i$. Since α is a limit ordinal, s_α is unbounded in α, but x_i is bounded below α for every $i < n$, we can certainly find a proper initial segment y of s_α such that $s_\alpha \cap x_i = y \cap x_i$, for every $i < n$. Let $\beta = \min(s_\alpha \setminus y)$ and $y' = y \cup \{\beta\}$. Then $y, y' \in X_\alpha$, $\phi(x, y) = \phi(x, y')$, but $\phi(s_\alpha, y') = \phi(s_\alpha, y) + 1$, and hence either $s_\alpha - x \not\perp y$ or $s_\alpha - x \not\perp y'$. We conclude that $s_\alpha - x \notin X_\alpha^\perp$. ∎

Claim 2. $\alpha \notin S \Rightarrow X_\alpha + X_\alpha^\perp = X$.

Proof of Claim 2. It suffices to prove $B \subset X_\alpha + X_\alpha^\perp$. Let $x \in B$. We may assume $\sup(x) \geq \alpha$, as otherwise $x \in X_\alpha$. Hence by property (2) for the coherent ladder system we have $\sup(x) > \alpha$ and $y = x \cap \alpha$ is bounded below α, hence $y \in X_\alpha$. We claim that $x - y \in X_\alpha^\perp$. It suffices to check for $z \in B \cap X_\alpha$ that $\phi(x, z) = \phi(y, z)$. So let $z \in B \cap X_\alpha$. Then $x \cap z = y \cap z$ and hence $\phi(x, z) = \phi(y, z)$ by definition of ϕ. ∎

Theorem 2.3. *Assume CH holds. Let $S \subset \omega_2$ be heavy and consisting of limit ordinals. Then there exists a coherent ladder system on S.*

Proof. Using CH, it is a standard procedure to construct a function $f\colon \omega_2 \to \mathscr{P}(\omega_2)$ with the following two properties:

(5) for every $\alpha < \omega_2$, for every countable $s \subset \alpha$, $s \in \operatorname{ran}(f | [\alpha, \alpha + \omega_1))$,

(6) for every $\alpha < \omega_2$, if $\omega_1 \beta < \alpha < \omega_1(\beta + 1)$ for some $\beta < \omega_2$, then for every finite $s \subset [\omega_1 \beta, \alpha)$, $s \in \operatorname{ran}(f|[\alpha, \alpha + \omega))$.

Note that *CH* is needed only for (5). Using f we define $\langle s_\alpha \colon \alpha \in S \rangle$ as follows:

Case 1. $\operatorname{cf}(\alpha) = \omega_1$. Since S is heavy we may choose club $C \subset \alpha \cap S$ with o.t. $C = \omega_1$. Suppose that an initial segment of s_α, say s has been defined. Let $\beta = \sup(s)$. So $\beta < \alpha$. By property (5) of f we may find $\gamma \in [\beta + 1, \alpha)$ such that $[\beta + 1, \gamma) \cap C \neq \phi$ and $f(\gamma) = s$. Then put γ into s_α. Proceeding similarly we ensure $\lim(s_\alpha) \subset S$, $s_\alpha \cap \gamma = f(\gamma)$ for every $\gamma \in s_\alpha$, and o.t.$(s_\alpha) = \omega_1$.

Case 2. $\operatorname{cf}(\alpha) = \omega$ and α is a limit of multiples of ω_1. Using again property (5) of f we construct s_α of order type ω such that $s_\alpha \cap \gamma = f(\gamma)$ for all $\gamma \in s_\alpha$.

Case 3. $\operatorname{cf}(\alpha) = \omega$ and $\omega_1 \beta < \alpha < \omega_1(\beta + 1)$ for some $\beta < \omega_2$. Utilize the method as in Case 2, but use property (6) of f for α' such that $\omega_1 \beta < \alpha' < \alpha$.

In all cases, the use of f ensures that the coherence property (3) holds. Property (4) is clear by the *CH*. ∎

COROLLARY 2.4. *Assume CH holds. For every heavy $S \subset \omega_2$ and every field k there exists a bilinear space (X, ϕ) over k of dimension $\aleph_2$ such that $\Gamma(X, \phi) = S$.*

Remark 2.5. Our construction above can be viewed as a generalization of Appenzeller's construction in [Ap].

Remark 2.6. The main result of [Sp] says that in every regular uncountable dimension there exist linearly topologized vector spaces which are linearly compact and do not have a continuous basis. The spaces witnessing this are Appenzeller's spaces in [Ap]. (The form naturally defines linear topologies.) Similar arguments as in [Sp] show that the spaces costructed above have these properties as well. (See [Sp] for the definitions.)

3. *CH* IS NOT NECESSARY FOR A FULL SPECTRUM

DEFINITION 3.1. The principle $\square_{\omega_1}$ says that there exists a family $\langle C_\alpha \colon \alpha \in \lim(\omega_2) \rangle$ such that

(i) $C_\alpha \subset \alpha$ is closed unbounded,

(ii) if $\operatorname{cf}(\alpha) = \omega$ then $|C_\alpha| \leq \omega$,

(iii) if $\gamma \in \lim(C_\alpha)$, then $C_\gamma = C_\alpha \cap \gamma$.

For sets C, D we will write $C \subset^* D$ if $C \setminus D$ is countable. C is called an *almost intersection* of a family $\langle C_\alpha : \alpha < \gamma \rangle$ if $C \subset^* C_\alpha$, for all $\alpha < \gamma$.

LEMMA 3.2. *Suppose that* $\square_{\omega_1}$ *holds and that for every family* $\langle D_\alpha :$ $\alpha < \omega_2 \rangle$ *of clubs on* ω_1 *there exists a club* $D \subset \omega_1$ *such that* $D \subset^* D_\alpha$ *for all* $\alpha < \omega_2$. *Then for every heavy* $S \subset \lim(\omega_2)$ *there exists a coherent ladder system on* S.

Proof. Let $\langle C_\alpha : \alpha \in \lim(\omega_2) \rangle$ be a $\square_{\omega_1}$-sequence. Fix a heavy $S \subset \lim(\omega_2)$. For $\alpha \in S \cap \operatorname{cof}(\omega_1)$, let $\langle \beta_i : i < \omega_1 \rangle$ enumerate the limit points of C_α increasingly. Since S is heavy we may choose $E_\alpha \subset \omega_1$ club such that $\beta_i \in S$ for all $i \in E_\alpha$. By assumption we can find a club $C \subset \lim(\omega_1)$ which is an almost intersection of $\langle E_\alpha : \alpha \in S \cap \operatorname{cof}(\omega_1) \rangle$. Let $C'_\alpha = \{ \beta_i :$ $i \in C \}$. Then clearly $C'_\alpha \subset^* S$ and if $\gamma \in C'_\alpha \cap C'_\beta$ then $C'_\alpha \cap \gamma = C'_\beta \cap \gamma$, since then γ is a limit point of C_α and C_β, hence by property (iii) of a $\square_{\omega_1}$-sequence $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

For $\alpha \in S \cap \operatorname{cof}(\omega_1)$ let s_α be the maximal tail of C'_α included in S. We claim that $\langle s_\alpha : \alpha \in S \cap \operatorname{cof}(\omega_1) \rangle$ has the coherence property (3). In fact, if $\gamma \in s_\alpha \cap s_\beta$ then $\gamma \in C'_\alpha \cap C'_\beta$ and hence $C'_\alpha \cap \gamma = C'_\beta \cap \gamma$ as shown above. Hence $s_\alpha \cap \gamma =$ maximal tail of $C'_\alpha \cap \gamma$ in $S =$ maximal tail of $C'_\beta \cap \gamma$ in $S = s_\beta \cap \gamma$.

Let $S_1 = \bigcup \{ \lim(s_\alpha) : \alpha \cap S \cap \operatorname{cof}(\omega_1) \}$. For $\alpha \in S_1$ pick $\beta \in S \cap \operatorname{cof}(\omega_1)$ with $\alpha \in \lim(s_\beta)$ and let $s_\alpha = s_\beta \cap \alpha$. We have just proved that s_α does not depend on the choice of s_β and hence that $\langle s_\alpha : \alpha \in S_1 \rangle$ is a coherent ladder system except for property (4) which will be proved below. We first extend $\langle s_\alpha : \alpha \in S_1 \rangle$ to all of S, as follows. Let $S_2 = S \setminus S_1$. Then $S_2 \subset \operatorname{cof}(\omega)$. We will do a similar construction as in the proof of Theorem 2.3 (but we do not use CH of course).

It is easy to construct in ZFC a function $f : \omega_2 \to \mathscr{P}(\omega_2)$ such that the following two properties hold.

(5′) for every $\alpha < \omega_2$, for every finite $s \subset \alpha$, $s \in \operatorname{ran}(f | [\alpha, \alpha + \omega_1) \cap \operatorname{succ})$

(6′) for every $\alpha < \omega_2$, if $\omega_1 \beta < \alpha < \omega_1(\beta + 1)$ for some $\beta < \omega_2$, then for every finite $s \subset [\omega_1 \beta, \alpha]$, $s \in \operatorname{ran}(f | [\alpha, \alpha + \omega) \cap \operatorname{succ})$.

Now working as in Theorem 2.3 we construct $\langle s_\alpha : \alpha \in S_2 \rangle$ a coherent lader system on S_2 such that $s_\alpha \subset \operatorname{succ}(\omega_2)$ has order type ω for all $\alpha \in S_2$. Since $s_\alpha \subset \lim(\omega_2)$ for all $\alpha \in S_1$, the system $\langle s_\alpha : \alpha \in S \rangle$ satisfies (1), (2), (3).

So it remains to prove (4). We prove it by induction on β. If $\beta \in \operatorname{succ}$ this is clear from the inductive assumption about $\beta - 1$. If $\beta \in \operatorname{cof}(\omega_1)$ we have

$$\{ s_\alpha \cap \beta : \alpha \in S \} \subset \bigcup \{ s_\alpha \cap \gamma : \gamma < \beta, \alpha \in S \} \cup \{ s_\beta \}$$

(where $\{s_\beta\}$ is omitted if $\beta \notin S$) and by the inductive assumption we are done. Finally, if $\beta \in \operatorname{cof}(\omega)$ we have

$$\{s_\alpha \cap \beta : \alpha \in S\} \subset \bigcup \{s_\alpha \cap \gamma : \gamma < \beta,\ \alpha \in S\}$$
$$\cup \{s_\alpha \cap \beta : s_\alpha \text{ is unbounded in } \beta\}.$$

But if s_α is unbounded in β then $\beta \in S$ and $s_\alpha \cap \beta = s_\beta$ by the coherence property. Hence by the inductive hypothesis we are done. ∎

In order to prove that CH is not necessary for the assertion of Theorem 2.3 to hold we will construct a model where the assumptions of Lemma 3.2 hold but CH fails. Let V be a model where CH and $\square_{\omega_1}$ hold, e.g., $V = L$.

DEFINITION 3.3. We define a forcing FC adding a "fast club" as follows. Its conditions are pairs (s, C) such that $s \subset \omega_1$ is countable and closed, and $C \subset \omega_1$ is a club. The ordering is defined by letting $(s, C) \leq (t, D)$ iff $t \subset s \subset t \cup (D \setminus \sup(t))$ and $C \subseteq D$.

It is easy to see that FC is σ-closed and if CH holds then FC is $\aleph_1$-centered. Moreover, if $G \subset FC$ is generic over V and $C_G = \bigcup\{s: (s, C) \in G\}$ then $C_G \subset^* C$ for every club $C \subset \omega_1$, $C \in V$. Hence if $\langle P_\alpha, Q_\beta : \alpha \leq \omega_3,\ \beta < \omega_3\rangle$ is a countable support iteration of FC, i.e., $Q_\alpha = FC^{V^{P_\alpha}}$ for all $\alpha < \omega_3$, and G in P_{ω_3}-generic over V, then P_{ω_3} has the $\aleph_2$-*c.c.*, P_{ω_3} is σ-closed and for every sequence $\langle C_\alpha : \alpha < \omega_2\rangle \in V[G]$ of clubs $C_\alpha \subset \omega_1$ there exists club $C \subset \omega_1$, $C \in V[G]$ such that C is an almost intersection of $\langle C_\alpha : \alpha < \omega_2\rangle$. Moreover, since P_{ω_3} is σ-closed and has the $\aleph_2$-*c.c.*, $V[G]$ has the same cardinals and cofinalities as V.

The proofs of these facts are standard and can be found in [B].

Since $\omega_2^V = \omega_2^{V(G)}$ and $V \models \square_{\omega_1}$, clearly $V[G] \models \square_{\omega_1}$. So the assumptions of Lemma 3.2 hold in $V[G]$. But of course $V(G) \models CH$ since P_{ω_3} in σ-closed.

Let Q be any *ccc* forcing which adds at least $\aleph_2$ new reals, e.g., the Cohen algebra for adding $\aleph_2$ Cohen reals. Let H be Q-generic over $V[G]$. We claim that $V[G][\![H]\!]$ is the desired model. It is enough to show that the assumptions of Lemma 3.2 are preserved in $V[G][\![H]\!]$. First, Q does not change cardinals or cofinalities, Hence $\omega_2^V = \omega_2^{V[G][\![H]\!]}$, and hence $V[G][\![H]\!] \models \square_{\omega_1}$. Moreover, it is well known (see [K]) that if P is a *ccc* forcing and $\dot{C}$ is a P-name for a club, there exists a club D in the ground model such that $\|D \subseteq \dot{C}\|_P = 1$. Hence given $\langle C_\alpha : \alpha < \omega_2\rangle$ a sequence of clubs $C_\alpha \subset \omega_1$ in $V[G][\![H]\!]$ we may find a sequence of clubs $\langle C_\alpha^1 : \alpha < \omega_2\rangle \in V[G]$ such that $C_\alpha^1 \subset C_\alpha$ for all $\alpha < \omega_2$. In $V[G]$ find club $C \subset \omega_1$ with $C \subset^* C_\alpha^1$ for all $\alpha < \omega_2$. Then $C \subset^* C_\alpha$, $\alpha < \omega_2$, in $V[G][\![H]\!]$. Hence we have proved the following theorem.

THEOREM 3.4. *It is consistent with the negation of CH that for every heavy* $S \subset \omega_2$, *for every field* k, *there exists a bilinear space* (X, ϕ) *over* k *of dimension* $\aleph_2$ *with* $\Gamma(X, \phi) = S$.

4. SPARSE SPECTRUM IN MITCHELL'S MODEL

In this section we assume the existence of a weakly compact cardinal in order to obtain an independence result on the spectrum of the Γ-invariant of bilinear spaces in dimension $\aleph_2$ over fields of arbitrary size.

In [M] Mitchell has constructed a model in which ω_2 has the tree property, i.e., every tree of height ω_2 and levels of size $< \aleph_2$ has a branch of length ω_2. We will prove that in Mitchell's model the assertion of Corollary 2.4 fails badly.

Let λ be a weakly compact cardinal. We define a forcing iteration $\langle P_\alpha, \dot{Q}_\beta\colon \alpha \leq \lambda, \beta < \lambda \rangle$ as follows. If $\alpha < \lambda$ is even then $\dot{Q}_\alpha = \mathbf{C}(\omega)$, where $\mathbf{C}(\omega)$ is the forcing which adds a Cohen subset of ω (with finite conditions). If $\alpha < \lambda$ is odd, then $\dot{Q}_\alpha = \mathbf{C}(\omega_1)^{V^{P_\alpha}}$, where $\mathbf{C}(\omega_1)$ is the forcing which adds one Cohen subset of ω_1 (with countable conditions). Conditions $p \in P_\alpha$ have mixed support, finite support on the even ordinals, and countable support on the odd ordinals, i.e., $|\mathrm{dom}(p) \cap \{\text{even ordinals}\}| < \aleph_0$ and $|\mathrm{dom}(p) \cap \{\text{odd ordinals}\}| \leq \aleph_0$.

It is well known that P_λ has the λ-*c.c.* (for this, the inaccessibility of λ is needed), P_λ is proper, and forcing with P_λ collapses every cardinal κ, $\omega_1 < \kappa < \lambda$, to ω_1. Hence $V^{P_\lambda} \models \omega_1 = \omega_1^V \wedge \omega_2 = \lambda$. Before stating the main result, let us recall the notion of a *weakly compact filter* associated with λ. A set $Y \subseteq \lambda$ belongs to it iff there exist $X \subseteq \lambda$ with $X \subseteq Y$, $R \subset V_\lambda$ (where V_λ is the set of sets of rank $< \lambda$), and a Π^1_1-formula ψ in one free second order variable such that $\langle V_\lambda, \in, R \rangle \models \phi(R)$ and X is the set of $\alpha < \lambda$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \models \phi(R \cap V_\alpha)$. It is a standard fact that the weakly compact filter is κ-complete and normal.

THEOREM 4.1. *Suppose* G *is* P_λ-*generic over* V *and in* V, $S \subset \lambda$ *is costationary and belongs to the weakly compact filter of* λ. *Then in* $V[G]$, *over no field* k *does there exist a bilinear space* (X, ϕ) *in dimension* $\aleph_2$ *such that* $\Gamma(X, \phi) = S$ *modulo club filter*.

Remark 4.2. Some explanation is needed to see why Theorem 4.1 together with Corollary 2.4 gives an independence result. For this it is enough to show that there exists a stationary costationary set S which belongs to the weakly compact filter of λ and is heavy in $V[G]$ (of Theorem 4.1). Note that such an S cannot be heavy in V. Let S be any set belonging to the weakly compact filter of λ (so S is stationary) such that S

contains all ordinals of cofinality ω and S is costationary. An example of such a set is the set containing all Mahlo cardinals below λ and all ordinals of cofinality ω. First, it is well known that a forcing with the λ-c.c. does not make stationary subsets of λ nonstationary. Hence, such S is stationary (and costationary) in $V[G]$. In $V[G]$, let $\kappa \in S$ be of uncountable cofinality. Then $\mathrm{cf}(\kappa) = \omega_1$. Hence there exists a club $C \subset \kappa$ of ordinals of cofinality ω. But since P_λ is proper, we have $C \subset \mathrm{cof}(\omega)^V$, and hence $C \subseteq S$. We have shown that S is heavy in $V[G]$.

In Theorem 4.1, the requirement that S be costationary cannot be dropped. Spaces in dimension $\aleph_2$ with Γ-invariant $S = \omega_2$ can be constructed in *ZFC*. Examples are the ω_1-Gross spaces in dimension $\aleph_2$ constructed in [ShSp]. These are spaces with the property that for every subspace of dimension at least $\aleph_1$, its orthogonal complement has dimension smaller than the dimension of the space.

The proof of Theorem 4.1 will be given first for the case that the field has size at most $\aleph_2$. The general case then will follow easily, using an argument with an elementary substructure of size $\aleph_2$.

Proof of Theorem 4.1 *for Fields of Size at Most* $\aleph_2$. Suppose $S \subset \lambda$, field k of size $\leq \aleph_2$, and the space (X, ϕ) over k were a counterexamaple to Theorem 4.1. We may certainly assume that the domain of k is an ordinal less than or equal to λ. Fix $\langle e_\alpha \colon \alpha < \lambda \rangle$ an algebraic basis of X.

DEFINITION 4.3. For $z \in X$ and $\alpha \leq \lambda$ we define the α-type of z as $t(z, \alpha) = \langle \phi(z, e_\beta) \colon \beta < \alpha \rangle$. Thus the α-type of z is an α-sequence of elements of $k \subseteq \lambda$.

Letting $X_\alpha = \mathrm{span}\langle e_\beta \colon \beta < \alpha \rangle$, it is clear that $\langle X_\alpha \colon \alpha < \lambda \rangle$ is a κ-filtration of X. Hence we know that $S = \{\alpha < \lambda \colon X_\alpha + X_\alpha^\perp \neq X\}$ modulo club filter. Therefore, without loss of generality we may assume that the two sets are equal. It is easy to see that then $\alpha \in S$ if and only if there exists $z_\alpha \in X$ such that no $x \in X_\alpha$ has the same α-type as z_α.

Since S belongs to the weakly compact filter of λ, in V we may find an inaccessible cardinal $\kappa \in S$ such that the following hold:

(a) for all $\alpha, \beta < \kappa$, $\phi(e_\alpha, e_\beta) \in k \cap \kappa$, and $k \cap \kappa$, if endowed with the restricted field operations, is a subfield of k and belongs to $V[G_\kappa]$;

(b) $V[G_\kappa] \vDash S \cap \kappa$ is stationary and costationary:

(c) in $V[G_\kappa]$ there exists a P_λ / G_κ-name $\dot{z}_\kappa$ for a vector in X such that

$$\|\text{no } x \in X_\kappa \text{ has the same } \kappa\text{-type as } \dot{z}_\kappa\|_{P_\lambda / G_\kappa} = 1.$$

Throughout the proof we let $z_\kappa = \dot{z}_\kappa[G_\lambda]$ and $r_\alpha = \phi(z_\kappa, e_\alpha)$. The subfield $k \cap \kappa$ of k is denoted by k'. Note that $r_\alpha \notin k'$, in general. However, if k has size $\leq \aleph_1$ in $V[G_\lambda]$, we may assume $k' = k$, $r_\alpha \subset k'$, and the proof is simplified.

LEMMA 4.4. *There exist finitely many $a_0, \ldots, a_{n-1} \in k$ such that for each $\alpha < \kappa$, $r_\alpha \in k'[a_0, \ldots, a_{n-1}]$, where $k'[a_0, \ldots, a_{n-1}]$ is the least subfield of k containing k' and $a_0, \ldots, a_{n-1}$.*

Proof of Lemma 4.4. Work in $V[G_\lambda]$. Viewing $k'[\{r_\alpha\colon \alpha < \kappa\}]$ as a k'-space, we show that it has finite dimension. Let $C \in [\kappa]^\omega$. We show that $\{r_\alpha\colon \alpha \in C\}$ is not linearly independent. Since in $V[G_\lambda]$, $\mathrm{cf}(\kappa) = \omega_1$ and $\kappa \setminus S$ is unbounded in κ, we may find $\alpha^* \in \kappa \setminus S$ so that $\alpha^* > \sup(C)$. Since $\alpha^* \notin S$ there exists $y^* \in X_{\alpha^*}$ such that $t(z_\kappa, \alpha^*) = t(y^*, \alpha^*)$. Let $y^* = \sum_{i<n} a_i e_{\beta_i}$, $\beta_i < \alpha^*$ for $i < n$. We conclude

$$\phi(y^*, e_\alpha) = \sum_{i<n} a_i \phi(e_{\beta_i}, e_\alpha) = r_\alpha,$$

for all $\alpha < \alpha^*$, so in particular for all $\alpha \in C$. Since for such α we have $\phi(e_{\beta_i}, e_\alpha) \in k'$, $\{r_\alpha\colon \alpha \in C\}$ is contained in the k'-span of $\{a_i\colon i < n\}$. Hence we may find $a_0, \ldots, a_{n-1}$ as desired. ∎

For the rest of the proof we fix $k'[a_0, \ldots, a_{n-1}]$ as in Lemma 4.4. Note that $k'[a_0, \ldots, a_{n-1}]$ does not contain any transcendental element over k', since the set of all powers of such an element would be infinite and linearly independent over k'. We conclude that $k'[a_0, \ldots, a_{n-1}]$ is obtained by adjoining to k' the roots of finitely many irreducible polynomials with coefficients in k'. It is therefore clear that in $V[G_\kappa]$ there exists a field k^* containing k', such that in $V[G_\lambda]$ there exists an isomorphism θ between $k'[a_0, \ldots, a_{n-1}]$ and k^* which is the identity on k'. For the rest of the proof we fix such k^* and θ.

For $\alpha < \lambda$ we will denote the k^*-space, $k'[a_0, \ldots, a_{n-1}]$-space with basis $\langle e_\beta;\ \beta < \alpha \rangle$ by $X_\alpha^*, \overline{X}_\alpha$, respectively. Since $\phi(e_\alpha, e_\beta) \in k'$ for all $\alpha, \beta < \kappa$, there is a unique way to extend the restriction of ϕ to the k^*-span of $\langle e_\alpha\colon \alpha < \kappa \rangle$ to a bilinear form ϕ^* on X_κ^*. For $x \in X_\kappa^*$ and $\alpha \leq \kappa$, by $t(x,\alpha)$ we denote $\langle \phi^*(x, e_\beta)\colon \beta < \alpha \rangle$. It is clear that θ induces an isomorphism between $\overline{X}_\alpha$ and X_α^* for all $\alpha < \lambda$, which we also denote by θ. For $x, y \in \overline{X}_\kappa$ we easily have $\theta\phi(x, y) = \phi^*(\theta x, \theta y)$. By abuse of notation, for $\alpha \leq \kappa$ we write $\theta t(x, \alpha)$ instead of $\langle \theta\phi(x, e_\beta)\colon \beta < \alpha \rangle$. Clearly we have $\theta t(x, \alpha) = t(\theta x, \alpha)$. We also use this notation if $x \notin \overline{X}_\kappa$, but $\phi(x, e_\beta) \in k'[a_0, \ldots, a_{n-1}]$ for all $\beta < \alpha$.

LEMMA 4.5. *For all $\alpha < \kappa$, $\theta t(z_\kappa, \alpha) \in V[G_\kappa]$.*

Proof of Lemma 4.5. In $V[G_\lambda]$, for every $\alpha \in \kappa \setminus S$ there exists a unique $y_\alpha \in X_\alpha$ such that $t(z_\kappa, \alpha) = t(y_\alpha, \alpha)$. We claim that $y_\alpha \in \overline{X}_\alpha$. In fact, let $y_\alpha = \sum_{i<m} b_i e_{\beta_i}$, $\beta_i < \alpha$ for $i < m$. We conclude

$$\phi(y_\alpha, e_\beta) = \sum_{i<m} b_i \phi(e_{\beta_i}, e_\beta) = r_\beta, \qquad \text{for all } \beta < \alpha.$$

This means that the system of equations

$$\sum_{i<m} Y_i \phi(e_{\beta_i}, e_\beta) = r_\beta, \qquad \beta < \alpha$$

which by Lemma 4.4 has coefficients in $k'[a_0, \ldots, a_{n-1}]$, has a solution (namely $(b_0, \ldots, b_{m-1})$) in the superfield k. Hence, by basic linear algebras, it has a solution in $k'[a_0, \ldots, a_{n-1}]$. Hence we obtain a vector in $\overline{X}_\alpha$ with the same α-type as z_κ. By uniqueness of y_α we conclude $y_\alpha \in \overline{X}_\alpha$. Consequently, we get $\theta y_\alpha \in V[G_\kappa]$, and hence $\theta t(z_\kappa, \alpha) = t(\theta y_\alpha, \alpha) \in V[G_\kappa]$. ∎

In $V[G_\kappa]$, we define a tree T as follows: T consists of all sequences $\langle r_\beta \colon \beta < \alpha \rangle$ such that $\alpha < \kappa$ and there exists a condition $p \in P_\lambda / G_\kappa$ (the forcing which extends $V[G_\kappa]$ to $V[G]$) such that

$$p \Vdash \dot{\theta} t(\dot{z}_\kappa, \alpha) = \langle r_\beta \colon \beta < \alpha \rangle.$$

The ordering of T is the extension of sequences. So T is a subtree of ${}^{<\kappa}k^*$.

By Lemma 4.5, $\theta t(z_\kappa, \kappa)$ is a branch through T. By the following lemma we conclude that $\theta t(z_\kappa, \kappa)$ is in $V[G_\kappa]$.

LEMMA 4.6 [M, Lemma 3.8]. *Suppose that* $\mathrm{cf}^V(\gamma) > \omega$, $t \colon \gamma \to V$ *is in* $V[G]$, *and* $t | \alpha \in V[G_\kappa]$ *for all* $\alpha < \gamma$. *Then t is in* $V[G_\kappa]$.

COROLLARY 4.7. $\theta t(z_\kappa, \kappa)$ *is in* $V[G_\kappa]$.

Working in $V[G_\lambda]$, by the proof of Lemma 4.5, for each $\alpha \in \kappa \setminus S$ there exists a unique $y_\alpha \in \overline{X}_\alpha$ such that $t(z_\kappa, \alpha) = t(y_\alpha, \alpha)$. Hence there exists a unique $y_\alpha^* \in X_\alpha^*$, namely θy_α, such that $t(y_\alpha^*, \alpha) = \theta t(z_\kappa, \alpha)$. Consequently, since $\theta t(z_\kappa, \kappa) \in V[G_\kappa]$, in $V[G_\kappa]$ we can define $\langle y_\alpha^* \colon \alpha \in \kappa \setminus S \rangle$. Since $\kappa \setminus S$ is stationary in $V[G_\kappa]$, and for limit α we have $y_\alpha^* \in \bigcup_{\beta<\alpha} X_\beta^*$, by Fodor's Lemma we may find a stationary $S' \subseteq \kappa \setminus S$ and $\alpha^* < \kappa$ such that $y_\alpha^* \in X_{\alpha^*}^*$ for all $\alpha \in S'$. But then by uniqueness of y_α^* we conclude that for $\alpha \in S' \setminus \alpha^*$, y_α^* is constant, say y^*.

We conclude $t(y^*, \kappa) = \theta t(z_\kappa, \kappa)$, and hence $t(\theta^{-1} y^*, \kappa) = t(z_\kappa, \kappa)$. Then $z_\kappa = \theta^{-1} y^* + (z_\kappa - \theta^{-1} y^*)$, $\theta^{-1} y^* \in X_{\alpha^*}$, and $z_\kappa - \theta^{-1} y^* \in X_\kappa^\perp$, i.e., $z_\kappa \in X_\kappa + X_\kappa^\perp$, a contradiction.

Proof of Theorem 4.1 *for Arbitrary Fields*. Suppose that $S \subseteq \omega_2$, field k of size $> \aleph_2$, and space (X, ϕ) over k were a counterexample to Theorem 4.1. Fix a basis $\langle e_\alpha \colon \alpha < \omega_2 \rangle$ of X, and let $X_\alpha = \operatorname{span}\langle e_\beta \colon \beta < \alpha \rangle$. We may assume $\Gamma(X, \phi) = S$.

In $V[G]$ choose an elementary substructure $(N, \in) \prec (H_\mu, \in)$ where $\mu = (2^{|k|})^+$, such that $|N| = \aleph_2$, $\omega_2 \subseteq N$, and $S, (X, \phi, k), \langle e_\alpha \colon \alpha < \omega_2 \rangle \in N$. Let $k' = k \cap N$, let X' be the k'-span of $\langle e_\alpha \colon \alpha < \omega_2 \rangle$, let X'_α, for $\alpha < \omega_2$, be the k'-span of $\langle e_\beta \colon \beta < \alpha \rangle$, and $\phi' = \phi | X' \times X'$. Then clearly we have $X' = X \cap N$ and $\phi' = \phi \cap N$, and hence ϕ' has values in k'. By elementarity, it is easy to see that (X', ϕ', k') is nondegenerate. We claim that $\Gamma(X', \phi', k') = \Gamma(X, \phi, k) = S$, which will contradict the first part of the proof of Theorem 4.1.

First, let $\alpha \in \omega_2$ be such that $X_\alpha + X_\alpha^\perp \neq X$. By elementarity, there exists $z \in X \cap N$ such that $z \notin X_\alpha + X_\alpha^\perp$. But then we have $z \in X'$, and we claim that $z \notin X'_\alpha + (X'_\alpha)^\perp$. If this is false, then $z = x + y$ for some $x \in X'_\alpha$ and $y \in (X'_\alpha)^\perp$. Then $x \in N$, and hence also $y = z - x \in N$. Moreover $N \models y \in X_\alpha^\perp$. By elementarity we have that $y \in X_\alpha^\perp$. But clearly $x \in X_\alpha$, and hence we have a contradiction.

Secondly, let $\alpha \in \omega_2$ be such that $X_\alpha + X_\alpha^\perp = X$, and let $z \in X'$ be arbitrary. Since $z \in X$, by elementarity, there exist $x \in X_\alpha \cap N$ and $y \in X_\alpha^\perp \cap N$ such that $z = x + y$. Then clearly we have that $x \in X'_\alpha$, and, since $X'_\alpha \subseteq X_\alpha$, $y \in (X'_\alpha)^\perp$. Hence $z \in X'_\alpha + (X'_\alpha)^\perp$. ▮

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